CONVERGENCE OF BEST RATIONAL TCHEBYCHEFF APPROXIMATIONS

BY

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1. **Introduction.** To any real-valued function f continuous on the interval $[a, b]$, there exists, for every pair of non-negative integers n, m a unique best Tchebycheff approximation $r^*(n, m)$ among all rational functions of the form

$$r(n, m; x) = \frac{a_0 + a_1 x + \cdots + a_n x^n}{b_0 + b_1 x + \cdots + b_m x^m},$$

in which the numerator and denominator polynomials are relatively prime and $\sum |b_i| = 1$; i.e., $r(n, m) \neq r^*(n, m)$ implies

$$E(n, m, f) = \max_{a \leq x \leq b} |f(x) - r^*(n, m; x)| < \max_{a \leq x \leq b} |f(x) - r(n, m; x)|.$$

Cf. Achieser [1].

In this paper, we investigate the behavior of the error functional $E(n, m, f)$, in terms of the continuity properties of the function f, for various hypotheses on the growth of the degrees n and m of the numerator and denominator polynomials. §2 generalizes some results of Walsh [2] concerning the conditions under which $E(n, m, f)$ approaches zero with increasing n and m, while §3 gives conditions under which $E(n, m, f)$ does not approach zero; the results in these sections are shown to have counterparts when other fundamental Markoff systems in $C[a, b]$ replace the system $\{1, x, x^2, \cdots\}$.

A theorem of Jackson [3] on Tchebycheff approximation by polynomials states that

$$E(n, 0, f) \leqq \frac{b - a}{2} \omega\left(f, \frac{\pi}{n}\right),$$

where ω is the modulus of continuity of f on $[a, b]$:

$$\omega(f, \delta) = \sup_{|x - y| \leqq \delta} |f(x) - f(y)|.$$

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§4 contains some extensions of this theorem to approximation by reciprocals of polynomials; §5 obtains a special estimate of $E(0,m,f)$ for the function $f(x) = |x|$.

If all that is known about a function to be approximated is that it is continuous, a theorem of Bernstein [4] shows that no upper bound can be established for the rapidity with which the error of the best polynomial Tchebycheff approximation approaches zero as the degree of the polynomial approaches infinity. §6 presents similar theorems for various classes of rational approximations.

2. **Convergence conditions.** In this section, we investigate the conditions under which the error functional $E(n,m,f)$ approaches zero with increasing n and m. The symbol $C[a,b]$ denotes the space of real-valued functions continuous on the interval $[a,b]$. Most of the possibilities are covered by the following theorem, which is implied directly by the Weierstrass theorem on polynomial approximation.

THEOREM. *If $f \in C[a,b]$, then*

$$\lim_{n\to\infty} E(n,m,f) = 0.$$

The remaining choices of interest are those for which n remains bounded by some constant k; these can be handled by studying the behavior of $E(k,m,f)$ for fixed k as m approaches infinity. The results for $E(k,m,f)$ can be found as corollaries to a more general theorem. A set of functions $\{g_1, g_2, \cdots\}$ is *fundamental* in $C[X]$, the space of all real-valued functions continuous on an arbitrary region X, if $C[X]$ is contained in the closure of the set of all finite linear combinations

$$g = \sum_{i=1}^{n} c_i g_i;$$

i.e., if any $f \in C[X]$ can be approximated arbitrarily well in the Tchebycheff sense by functions of the form $\sum^n c_i g_i$.

THEOREM 1. *If the set of functions $\{g_1, g_2, \cdots\}$ is fundamental in $C[X]$, and if $f \in C[X]$ does not change sign on X, then f can be approximated arbitrarily well in the Tchebycheff sense by a function of the form*

$$\frac{1}{\sum_{i=1}^{n} c_i g_i}.$$

Proof. Assume $f(x) \geqq 0$, without loss of generality, and define

$$f_N(x) = \begin{cases} 1/f(x) & \text{if } |f(x)| \geqq 1/N, \\ N & \text{if } |f(x)| < 1/N. \end{cases}$$

As $f \in C[X]$, so is f_N for any N. For any function of the form

$$g = \sum_{i=1}^{n} c_i g_i,$$

it follows that

$$\left\| f - \frac{1}{g} \right\|_T \leqq \left\| f - \frac{1}{f_N} \right\|_T + \left\| \frac{1}{f_N} - \frac{1}{g} \right\|_T$$

$$\leqq \frac{1}{N} + \left\| \frac{g - f_N}{g f_N} \right\|_T .$$

Choose any $\epsilon, 0 < \epsilon < 1$, and let $N > 2/\epsilon$. Since $\{g_1, g_2, \cdots\}$ is fundamental in $C[X]$, there exists a function of the form g such that

$$\| f_N - g \|_T < \frac{\epsilon}{B(2B+1)},$$

where $B = \|f\|_T$. As $f_N(x) \geqq 1/B$,

$$|g(x)| > \frac{1}{B} - \frac{\epsilon}{B(2B+1)} > \frac{1}{B} - \frac{1}{B(2B+1)} = \frac{2}{2B+1}.$$

Thus,

$$\left\| f - \frac{1}{g} \right\|_T \leqq \frac{1}{N} + \left\| \frac{1}{f_N} \right\|_T \cdot \left\| \frac{1}{g} \right\|_T \cdot \| g - f_N \|_T$$

$$\leqq \frac{\epsilon}{2} + B \cdot \frac{2B+1}{2} \cdot \frac{\epsilon}{B(2B+1)} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Theorem 1 also holds if $C[X]$ is taken to be the space of all complex-valued functions continuous on a region X. When applied to rational approximation of functions analytic in the interior of a closed Jordan region in the complex plane and continuous on the boundary, the complex form of Theorem 1 and a result of Walsh [5], guaranteeing that the error of polynomial approximations to such functions approaches zero, imply the following theorem, also due to Walsh [2].

THEOREM (WALSH). *If f is analytic and different from zero in the interior of a closed Jordan region X in the complex plane, and continuous on the boundary of X, then*

$$\lim_{m \to \infty} E(0, m, f) = 0.$$

Theorem 1 and the Weierstrass theorem, or the above Walsh theorem and the Cauchy integral theorem, imply the corresponding result for continuous functions on the interval $[a, b]$.

COROLLARY 1a. *If $f \in C[a, b]$ and f does not change sign on $[a, b]$, then*

$$\lim_{m\to\infty} E(0,m,f) = 0.$$

Theorem 1 implies a number of stronger results, of which the following is typical. If $\{p_i\}$ is a sequence of positive real numbers such that

$$\sum_{i=1}^{\infty} \frac{1}{p_i} = \infty, \quad \lim p_i = \infty,$$

and $X = [a,b]$, then, by a theorem of Muntz [6], the set of functions $\{1, x^{p_1}, x^{p_2}, \cdots\}$ is fundamental in $C[a,b]$. Thus follows

COROLLARY 1b. *If $f \in C[a,b]$ and f does not change sign on $[a,b]$, and the sequence $\{p_i\}$ is as above, then f can be approximated arbitrarily closely in the Tchebycheff sense by functions of the form*

$$g(x) = \frac{1}{a_0 + a_1 x^{p_1} + \cdots + a_m x^{p_m}}.$$

A further remark on Theorem 1 is that if X is not connected, it is allowable for f to have opposite signs on different components of X, as long as the change of sign does not occur in the component.

Another corollary gives a result for $E(k,m,f)$ for a fixed integer k.

COROLLARY 1c. *If $f \in C[a,b]$ with sign changes at $x_1, \cdots, x_j \in [a,b]$, with $j \leqq k$, and $f(x)/(x-x_1)\cdots(x-x_j)$ is continuous on $[a,b]$, then*

$$\lim_{m\to\infty} E(k,m,f) = 0.$$

Proof. Let $P_j = (x-x_1)\cdots(x-x_j)$. Then, for any polynomial P_m of degree m,

$$\left\| f - \frac{P_j}{P_m} \right\|_T \leqq \|P_j\|_T \cdot \left\| \frac{f}{P_j} - \frac{1}{P_m} \right\|_T.$$

Since $f/P_j \in C[a,b]$ and does not change sign on $[a,b]$, Corollary 1a guarantees that

$$\lim_{m\to\infty} E(0,m,f/P_j) = 0.$$

Since

$$E(k,m,f) \leqq \|P_j\|_T E(0,m,f/P_j),$$

it follows that

$$\lim_{m\to\infty} E(k,m,f) = 0.$$

3. **Results of negative character.** Some of the above results for $E(k,m,f)$ have converses—i.e., it can be shown that continuous functions which

change sign too many times cannot be approximated uniformly well by functions of form $r(k, m)$ for fixed k. A general result of this nature can be stated in terms of a *Markoff system* $\{g_1, g_2, \cdots\}$: a system of functions in $C[a,b]$ such that no nontrivial generalized polynomial $c_1g_1 + \cdots + c_ng_n$ formed from the first n functions in the system vanishes at more than n points.

THEOREM 2. *If $\{g_1, g_2, \cdots\}$ is a Markoff system in $C[a,b]$, and $f \in C[a,b]$ changes sign more than k times in $[a,b]$, then f cannot be approximated arbitrarily well in the Tchebycheff sense by functions of the form*

$$g = \frac{c_1g_1 + \cdots + c_kg_k}{d_1g_1 + \cdots + d_mg_m}.$$

Proof. Suppose f changes sign at $\tilde{x}_1, \cdots, \tilde{x}_{k+1}$; then f alternates at some $k+2$ points $x_1, \cdots, x_{k+2} \in [a,b]$. Let

$$\eta = \min(|f(x_1)|, \cdots, |f(x_{k+2})|).$$

If there exists

$$g = \frac{c_1g_1 + \cdots + c_kg_k}{d_1g_1 + \cdots + d_mg_m}.$$

such that $\|f - g\|_T < \eta$, then g also alternates at $x_1, \cdots, x_{k+2}$. However, the only way g can change sign in $[x_i, x_{i+1}]$ and satisfy $\|f - g\|_T < \eta < \infty$ is for $\sum^k c_jg_j(\bar{x}_i) = 0$ for some $\bar{x}_i \in [x_i, x_{i+1}]$, for $i = 1, \cdots, k+1$. But, since $\{g_1, g_2, \cdots\}$ is a Markoff system, this implies $\sum^k c_jg_j = 0$, contradicting the statement $\|f - g\|_T < \eta$.

Theorem 2 thus provides converses to Corollaries 1a and 1b.

COROLLARY 2a. *If $f \in C[a,b]$, then*

$$\lim_{m \to \infty} E(0, m, f) = 0$$

if and only if f does not change sign on $[a,b]$.

COROLLARY 2b. *If $f \in C[a,b]$ and $a > 0$, then f can be approximated arbitrarily well on $[a,b]$ by functions of the form*

$$g(x) = \frac{1}{a_0 + a_1x^{p_1} + \cdots + a_mx^{p_m}},$$

for integers $p_i > 0$ such that

$$\sum_{i=1}^{\infty} \frac{1}{p_i} = \infty, \quad \lim_{i \to \infty} p_i = \infty,$$

if and only if f does not change sign on $[a,b]$.

Proof. This follows from Corollary 1b and Theorem 2, since the functions x^{p_i}, $i = 1, 2, \cdots$, form a Markoff system on the positive real axis.

A partial converse to Corollary 1c is also provided by Theorem 2.

COROLLARY 2c. *If $f \in C[a, b]$, and f changes sign more than k times on $[a, b]$, then*

$$\lim_{m \to \infty} E(k, m, f) \neq 0.$$

For a function f satisfying a Lipschitz condition with constant M, the Jackson theorem states:

$$f \in \operatorname{Lip} M \Longrightarrow E(n, 0, f) \leqq \frac{b - a}{2} \cdot \frac{\pi M}{n}.$$

It has been conjectured that perhaps a stronger result holds for rational approximation—i.e., perhaps

$$f \in \operatorname{Lip} M \Longrightarrow E(n, m, f) \leqq \frac{cM}{n + m}$$

for some constant c. Corollary 2c shows that this is not the case; in fact, we have the following statements.

COROLLARY 2d. *If $F \subset C[a, b]$ includes a function which changes sign more than k times on $[a, b]$, and if $g(n, m, f)$ is a functional such that*

$$\lim_{m \to \infty} g(k, m, f) = 0$$

for all $f \in F$, then $g(n, m, f)$ is not an upper bound for $E(n, m, f)$ for the family F.

COROLLARY 2e. *If we define, for a family $F \subset C[a, b]$, the functional*

$$E(n, m, F) = \sup_{f \in F} E(n, m, f),$$

then there exists no constant c such that

$$E(n, m, \operatorname{Lip} M) \leqq \frac{cM}{n + m}$$

for all n, m.

It should be noted, however, that Theorem 2 does not hold in the space of complex-valued functions continuous on $[a, b]$. Any real-valued function on $[a, b]$, regardless of the number of sign changes, can be approximated arbitrarily well by the reciprocal of a complex-valued polynomial not vanishing on $[a, b]$. For, given any real-valued $f \in C[a, b]$ and any $\epsilon > 0$, there is some compact set X in the complex plane, containing $[a, b]$, for which the function $f_i = f + i\epsilon/2$ does not vanish. By the complex form of Theorem 1, then, there exists a polynomial P_i such that $|f_i(z) - 1/P_i(z)|$

$< \epsilon/2$ for $z \in X$. Therefore, on $[a,b]$, the polynomial P_i does not vanish and, furthermore, satisfies

$$\left\| f - \frac{1}{P_i} \right\|_T \leqq \|f - f_i\|_T + \left\| f_i - \frac{1}{P_i} \right\|_T < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

4. The effect of continuity properties on the error of the best rational Tchebycheff approximation. In this section, we consider extensions to rational approximation of the Jackson theorem, stated in terms of the modulus of continuity $\omega(f,\delta)$ of a function f defined on $[-1,1]$:

$$E(n,0,f) \leqq \omega\left(f, \frac{\pi}{n+1} \right).$$

It was shown in the previous section that the Jackson theorem cannot be extended to $E(n,m,f)$ in such a way as to obtain

$$\lim_{m \to \infty} E(k,m,f) = 0$$

for fixed k and arbitrary $f \in \mathrm{Lip}\, M$; this is because $\mathrm{Lip}\, M$ contains functions which change sign more than k times on $[-1,1]$. However, we can use the Jackson results to obtain upper bounds on the error functional $E(k,m,f)$ for certain subclasses of function f which do not change sign on $[-1,1]$, in particular, the family

$$F_\rho = \{f : |f(x)| \geqq \rho\}.$$

We begin with the following simple lemma.

Lemma 1. *If $f \in F_\rho$, then*

$$\omega\left(\frac{1}{f}, \delta \right) \leqq \frac{\omega(f,\delta)}{\rho^2}.$$

Proof.

$$\omega\left(\frac{1}{f}, \delta \right) = \sup_{|x-y| \leqq \delta} \left| \frac{1}{f(x)} - \frac{1}{f(y)} \right|$$

$$= \sup_{|x-y| \leqq \delta} \left| \frac{f(y) - f(x)}{f(x)f(y)} \right| \leqq \frac{\omega(f,\delta)}{\rho^2}.$$

The following theorem and its corollary give estimates of $E(0,m,f)$ and $E(k,m,f)$ for suitably restricted functions f.

Theorem 3. *If $f \in F_\rho \cap C[-1,1]$, then, for every $\epsilon > 0$, there exists N such that $m > N$ implies*

$$E(0,m,f) < (1+\epsilon) \cdot \frac{\|f\|_T^2\, \omega(f, \pi/(m+1))}{\rho^2}.$$

Proof.

$$\left\| f - \frac{1}{P_m} \right\|_T = \left\| \frac{1}{1/f} - \frac{1}{P_m} \right\|_T = \left\| \frac{P_m - 1/f}{P_m \cdot 1/f} \right\|_T.$$

By the Jackson theorem and Lemma 1,

$$\left\| f - \frac{1}{P_m} \right\|_T \leqq \left\| \frac{1}{P_m} \right\|_T \cdot \left\| \frac{1}{1/f} \right\|_T \cdot \omega\left(\frac{1}{f}, \frac{\pi}{m+1}\right)$$

$$\leqq \left\| \frac{1}{\dfrac{1}{f} - \dfrac{\omega(f, \pi/(m+1))}{\rho^2}} \right\|_T \|f\|_T \cdot \frac{\omega(f, \pi/(m+1))}{\rho^2}.$$

Since $\omega(f, \pi/(m+1)) \downarrow 0$ as $m \to \infty$, it follows, for any $\epsilon > 0$, that, for m sufficiently large,

$$\left\| f - \frac{1}{P_m} \right\|_T < (1+\epsilon) \cdot \frac{\|f\|_T^2 \, \omega(f, \pi/(m+1))}{\rho^2}.$$

COROLLARY 3a. *If, for some polynomial P_k, $f/P_k \in F_\rho \cap C[a,b]$, then, for every $\epsilon > 0$, there exists N such that $m > N$ implies*

$$E(k,m,f) < (1+\epsilon) \cdot \frac{\|f\|_T \cdot \|f/P_k\|_T \cdot \omega(f/P_k, \pi/(m+1))}{\rho^2}.$$

Proof. This statement follows from Theorem 3 and the relations

$$\left\| f - \frac{P_k}{P_m} \right\|_T = \left\| \frac{P_m f - P_k}{P_m} \right\|_T \leqq \|f\|_T \cdot \left\| \frac{1}{P_m} \right\|_T \cdot \left\| P_m - \frac{P_k}{f} \right\|_T.$$

Theorem 3 and Corollary 3a can be extended by using extensions of the Jackson theorem; for example, the following corollary holds.

COROLLARY 3b. *If $f, f', \cdots, f^{(p)} \in F_\rho \cap C[a,b]$, and $f^{(p)} \in \mathrm{Lip}\,\beta, 0 < \beta \leqq 1$, then, for some constant M,*

$$E(0,m,f) \leqq \frac{M}{m^{p+\beta}}.$$

As a final remark in this section, we observe that the bounds on $E(n,m,f)$ implied by the above results are no stronger than those given by the Jackson theorem for $E(n,0,f)$. Approximation of the function $f+c$ by $1/P_m$ is equivalent to approximation of f by

$$\frac{1}{P_m} - c = \frac{1 - cP_m}{P_m} = \frac{\widetilde{P}_m}{P_m};$$

thus $E(m,m,f) \leqq E(0,m,f+c)$ for any constant c. Since as c becomes large, the ratio $\|f\|^2/\rho^2$ approaches unity, it follows by Theorem 3 for every $\epsilon > 0$ that for sufficiently large m,

$$E(m,m,f) < (1+\epsilon)\omega\left(f,\frac{\pi}{m+1}\right)$$

which is equivalent to the conclusion obtained by the Jackson theorem for all m.

5. **An estimate of** $E(0,m,f)$ **for** $f(x)=|x|$. There are some functions $f \in C[-1,1]$, such as $f(x)=|x|$, which are not in F_ρ for any ρ nor do they satisfy $f/P_k \in F_\rho \cap C[-1,1]$ for any P_k, so that neither Theorem 3 nor Corollary 3a apply. However, they have no sign change on $[-1,1]$, so that it is known by Corollary 1a that

$$\lim_{m\to\infty} E(0,m,f) = 0.$$

Sometimes error estimates can be obtained for such functions by special devices; we proceed to estimate $E(0,m,f)$ for $f(x)=|x|$ on $[-1,1]$.

Define

$$f_m(x) = \begin{cases} \dfrac{1}{|x|}, & |x| \geqq m^{-1/3}, \\ m^{1/3}, & |x| < m^{-1/3}. \end{cases}$$

Since $f_m(x) \in C[-1,1]$, and, since

$$\left\| f - \frac{1}{P_m} \right\|_T \leqq \left\| f - \frac{1}{f_m} \right\|_T + \left\| \frac{P_m - f_m}{P_m f_m} \right\|_T,$$

it follows that

$$\begin{aligned} E(0,m,f) &\leqq m^{-1/3} + \left\| \frac{1}{P_m f_m} \right\|_T \omega\left(f_m, \frac{\pi}{m+1}\right) \\ &\leqq m^{-1/3} + \frac{1}{1-\epsilon_m}\,\frac{\pi}{m+1} \sup_{[-1,1]} f_m'(x), \end{aligned}$$

where $\epsilon_m \downarrow 0$ with increasing m. Thus, for m sufficiently large,

$$E(0,m,f) \leqq m^{-1/3} + \frac{\pi}{m} m^{2/3} = \frac{1+\pi}{m^{1/3}}.$$

6. **Lethargy theorems for rational Tchebycheff approximations.** In the previous section, we obtained estimates of the error incurred in approximating a function $f \in C[a,b]$ by rational functions, in terms of certain smoothness properties enjoyed by the function in addition to its continuity. Suppose all we know is that $f \in C[a,b]$. Can we find any estimates of the rapidity with which the error $E(n,m,f)$ approaches zero as n and m approach infinity? This question has been answered by Bernstein [4] in the negative for the error $E(n,0,f)$ of the best polynomial approximation.

THEOREM (BERNSTEIN). *For any nonincreasing sequence of real numbers* $\{A_n\}$ *with limit zero, there exists* $f \in C[a,b]$ *such that*

$$E(n,0,f) = A_n.$$

Theorems of this sort have been termed lethargy theorems, as they state that $E(n,0,f)$ can approach zero with any prescribed degree of lethargy as n approaches infinity. Some approximation properties of the partial sums of certain series of Tchebycheff polynomials allow us to obtain lethargy theorems for rational Tchebycheff approximations. The following lemma, taken from [7], is an extension of a result of Bernstein [8].

LEMMA 2. 1. *For any summable sequence* $\alpha = \{a_i\}$ *of nonnegative real numbers* a_i, *the function*

$$f(\alpha,k,q) = \sum_{i=1}^{\infty} a_i T_{kq^i-1},$$

where T_j *is the Tchebycheff polynomial of degree* j *shifted to the interval* $[a,b]$, k *is a positive integer, and* q *is an odd integer greater than unity, is continuous on* $[a,b]$.

2. *For any* $n \geqq 0$, *the polynomial*

$$f_n(\alpha,k,q) = \sum_{i=1}^{n} a_i T_{kq^i-1}$$

has, as an approximation to f, *an error curve*

$$f(\alpha,k,q) - f_n(\alpha,k,q) = \sum_{i=n+1}^{\infty} a_i T_{kq^i-1}$$

which alternates in sign $k \cdot q^n + 1$ *times on* $[a,b]$ *with common amplitude*

$$A_n = \sum_{i=n+1}^{\infty} a_i.$$

3. *Thus, as a best approximation to* $f(\alpha,k,q)$,

$$f_n(\alpha,k,q) = r^*(kq^{n-1} + t, u)$$

for $n \geqq 1$ *and for all integers* t, u *such that*

$$0 \leqq t \leqq k(q^n - q^{n-1}) - 1, \quad 0 \leqq u \leqq k(q^n - q^{n-1}) - 1.$$

Also, the constant 0 *is the best approximation*

$$r^*(t,u) \quad \text{for } 0 \leqq t \leqq k-1, \qquad 0 \leqq u \leqq k-1.$$

Note that if the sequence α consists of powers of a positive real number $a < 1$, the resulting function $f(\alpha,1,q)$ is an example of the celebrated Weierstrass function: if $aq \geqq 1$, $f(\alpha,1,q)$ is continuous on $[-1,1]$ but nondifferentiable there (cf. Achieser [1]). By proper choice of the integers k and q and the sequence α, we then obtain the following lethargy theorems.

THEOREM 4. *For any nonincreasing sequence of real numbers $\{A_n\}$ with limit zero and for any fixed integer k, there exists $f \in C[a,b]$ such that*

$$E(k \cdot 3^{n-1}, k-1, f) = A_n.$$

Proof. Consider the function

$$f(\alpha, k, 3) = \sum_{i=1}^{\infty} a_i T_{k \cdot 3^{i-1}},$$

where $\alpha = \{a_i\}$ is such that $a_i = A_{i-1} - A_i$, and T_j is the Tchebycheff polynomial of degree j shifted to $[a,b]$. By Lemma 2, the polynomial

$$f_n(\alpha, k, 3) = \sum_{i=1}^{n} a_i T_{k \cdot 3^{i-1}}$$

is the best approximation $r^*(k \cdot 3^{n-1}, k-1)$, with an associated approximation error

$$E(k \cdot 3^{n-1}, k-1, f) = \sum_{i=n+1}^{\infty} a_i = A_n.$$

COROLLARY 4a. *For any nonincreasing sequence of real numbers $\{A_n\}$ with limit zero, any fixed integer k, and any sequence of integers $\{j_n\}$ such that $j_{n+1} \geqq 3j_n$, there exists $f \in C[a,b]$ such that*

$$E(j_n, k-1, f) = A_n.$$

Proof. Let $m_n = \log_3(j_n/k)$; each j_n produces a different integer m_n, since $j_{n+1} \geqq 3j_n$. Then,

$$f(\alpha, k, 3) = \sum_{i=1}^{\infty} a_i T_{k \cdot 3^{m_i}},$$

where $a_i = A_{i-1} - A_i$, satisfies the claim. This follows since, by Lemma 3,

$$E(k \cdot 3^{m_n}, k-1, f) = \cdots = E(k \cdot 3^{m_{n+1}} - 1, k-1, f) = A_n,$$

and since, by construction, each j_n is between $k \cdot 3^{m_n}$ and $k \cdot 3^{m_{n+1}} - 1$.

The following lethargy theorem holds for all families of rational functions in which the ratio of the degree of the denominator polynomial to the degree of the numerator polynomial is bounded by some nonnegative real number c.

THEOREM 5. *For any nonincreasing sequence of real numbers $\{A_n\}$, with limit zero, and any real number $c > 0$, there exists $f \in C[a,b]$ such that, for some infinite sequence $\{j_n\}$,*

$$E(j_n, [cj_n], f) = A_n.$$

Proof. Consider the function

$$f = \sum_{i=1}^{\infty} (A_{i-1} - A_i) T_{3^{k_i}},$$

where the k_i satisfy

$$\frac{3^{k_i+1} - 3^{k_i} - 1}{3^{k_i}} > c.$$

Since Lemma 3 guarantees that

$$E(3^{k_n}, 3^{k_n+1} - 3^{k_n} - 1, f) = A_n,$$

the choice $j_n = 3^{k_n}$ satisfies

$$E(j_n, [cj_n], f) = A_n.$$

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